Resit — Group Theory (WIGT-07)

Thursday April 12, 2018, 9:00h–12.00h

University of Groningen

Instructions

1. Write your name and student number on every page you hand in.
2. All answers need to be accompanied with an explanation or a calculation.
3. Your grade for this exam is $(P + 10)/10$, where P is the number of points for this exam.

Problem 1 (15 points)

a) Give the definition of the order of an element of a group.

Solution: Let G be a group and $x \in G$. Then the order of x is the smallest positive integer n such that x^n is the unit element of G, if such an integer exists. (3 points) Otherwise it is defined to be ∞. (2 points)

b) Write down Cayley's theorem.

Solution: Every group G is isomorphic to a subgroup of S_G. (5 points) If $\#G = n$ is finite, then G is isomorphic to a subgroup of S_n.

c) Let G be a finite group and let p be a prime dividing $\#G$. Give the definition of a Sylow-p group in G.

Solution: Let $\#G = p^n \cdot m$, where $n \geq 1$ and $\gcd(p, m) = 1$. Then a Sylow p-group in G is a subgroup of G of order p^n. (5 points)

Problem 2 (10 points)

What is the number of subgroups of S_5 of order 5?

Solution: We have $\#S_5 = 120 = 5 \cdot 24$. So the number of subgroups of S_5 of order 5 is the number N_5 of 5-Sylow groups in S_5. By Sylow's theorem, $N_5 \equiv 1 \pmod 5$ and $N_5 \mid 24$, so $N_5 \in \{1, 6\}$. (3 points) However, $N_5 = 1$ is impossible, because every 5-cycle has order 5, so it generates a subgroup of order 5, and there are too many (namely 24) 5-cycles to all lie in the same group. (7 points) Explicitly, $(1\,2\,3\,5\,4)$ is not in the group generated by $\sigma = (1\,2\,3\,4\,5)$, since the only element of the latter sending 1 to 2 is σ. Hence $N_5 = 6$.

Alternatively, we can show this directly without appealing to Sylow's theorem, by noting that any subgroup of order 5 must be generated by a 5-cycle (5 points) and using that each of them contains exactly 4 5-cycles, so there are $24/4 = 6$ of them. (5 points)

Problem 3 (20 points)

Let G be a group and let H be a subgroup of G. Define

$$N := \bigcap_{a \in G} aHa^{-1}$$

and

$$X := \{aH : a \in G\}.$$

a) Show that $N \subseteq H$. (1 point)

Solution: If $x \in N$, then $x \in eHe^{-1} = H$. (1 point)

b) Show that N is a subgroup of G. (4 points)

Solution: For $a \in G$, we have $e = aea^{-1} \subseteq aHa^{-1}$. (1 point) Now let $x, x' \in N$ and let $h, h' \in H$ such that $x = aha^{-1}$ and $x' = ah'a^{-1}$. Then $xx' = aha^{-1}ah'a^{-1} = ahh'a^{-1} \subseteq aHa^{-1}$, (1 point) since H is a subgroup. For the same reason $x^{-1} = aha^{-1} \subseteq aHa^{-1}$. (1 point) Hence N is a subgroup of G by the subgroup criterion. (1 point)

c) Show that N is a normal subgroup of G. (5 points)

Solution: We'll show $bNb^{-1} \subseteq N$ for all $b \in G$, which is equivalent to N being normal. (1 point) So let $x \in N$ and $y = bxb^{-1}$. Let $a \in G$, then we need to show that $y = aha^{-1}$ for some $h \in H$. As $x \in N$, we can find $h \in H$ such that $x = (b^{-1}a)h(b^{-1}a)^{-1}$ (3 points). But then $y = b(b^{-1}aha^{-1}b)b^{-1} = aha^{-1}$ as desired (1 point).

d) For $x \in G$, let $\varphi_x : X \to X$ denote the map $aH \mapsto xaH$. Show that φ_x is a bijection for all $x \in G$, and that the map $\Phi : G \to S_X$ given by $x \mapsto \varphi_x$ is a homomorphism, where S_X is the group of bijections on X. (3 points)

Solution: It's obvious that $\varphi_{xy} = \varphi_x \circ \varphi_y$ for all $x, y \in G$. (1 point) In particular, this shows that $\varphi_{x^{-1}}$ is an inverse function of φ_x, so φ_x is a bijection. (1 point) The fact that Φ is a homomorphism has already been proved (1 point).

e) Suppose that the index $[G : H]$ is finite. Use d) to prove that $[G : N]$ divides $[G : H]!$. (7 points)

Solution: Since $\#X = [G : H]$ by definition, we find by the first isomorphism theorem (1 point) and Lagrange's theorem (1 point)

$$[G : \ker(\Phi)] = \#(G/\ker(\Phi)) = \#\Phi(G) \mid \#S_X = [G : H]!$$

(2 points) To show the result, it therefore suffices to prove that $\ker(\Phi) \subseteq N$ (in fact we have equality) (1 point). So let $x \in \ker(\Phi)$ and $a \in G$. Then $xaH = aH$, so there are $h, h' \in H$ such that $xah = ah'$. (1 point) Therefore $x = a(h'h^{-1})a^{-1} \in aHa^{-1}$, so $x \in N$ (1 point).

Problem 4 (10 points)

Show that there is no simple group of order 351.

Solution: Let G be a group of order $351 = 3^3 \cdot 13$. (1 point) For a prime $p \mid 351$, let N_p be the number of Sylow-p groups in G. If we find $N_p = 1$ for some p, then we know that the unique Sylow p-group in G is normal and, since it has order p, it is not G or $\{e\}$, so G is not simple. (2 points) By Sylow's theorem $N_{13} \equiv 1 \pmod 13$ and $N_{13} \mid 27$, so $N_{13} \in \{1, 27\}$. Suppose $N_{13} = 27$; it suffices to show that $N_3 = 1$. (2 points) If H, H' are distinct 13-Sylow groups in G, then their intersection consists only of the unit element e (for instance, since their intersection is a subgroup of H, so by Lagrange it has order dividing 13). Hence there are $27 \cdot (13 - 1)$ elements of order 13 in G, as every element $\neq e$ of a 13-Sylow group has order 13. (3 points) This implies $N_3 = 1$, because $N_3 \geq 1$ and every Sylow-3 group consists of precisely 27 elements, none of which are of order 13. (2 points)

Problem 5 (15 points)

Let $\mathbb{T}$ denote the set of complex numbers of absolute value 1.

a) Show that $\mathbb{T}$ is an abelian group under multiplication.

Solution: We show that $\mathbb{T}$ is a subgroup of the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ (1 point) (which is abelian, so that $\mathbb{T}$ is automatically abelian as well (1 point)). Clearly $1 \in \mathbb{T}$ and, since $|zw| = |z||w|$ for all $z, w \in \mathbb{C}$, we have $zw \in \mathbb{T}$ and $z^{-1} \in \mathbb{T}$ for $z, w \in \mathbb{T}$. (3 points) Alternatively, one can use that the absolute value defines a homomorphism $\mathbb{C}^\times \to \mathbb{R}_{>0}$ with kernel $\mathbb{T}$.

b) Show that $\mathbb{T}$ is isomorphic to $\mathbb{R}/2\pi\mathbb{Z}$.

Solution: The map $\theta \mapsto \exp(i\theta)$ (2 points) defines a homomorphism from $\mathbb{R}$ to $\mathbb{T}$, since $\exp(i(\theta + \theta')) = \exp(i\theta)\exp(i\theta')$ by properties of the exponential map (2 points). The map is clearly surjective (for given $z \in \mathbb{T}$, take $\theta \in [0, 2\pi)$ to be the angle of z) (2 points), and the kernel is equal to $2\pi\mathbb{Z}$ (1 point), so the result follows from the first isomorphism theorem (3 points).

Problem 6 (20 points)

Let

$$H := \{(a, b, c) \in \mathbb{Z}^3 \,:\, a + 2b + 4c \equiv 0 \pmod 8\}$$

a) Show that H is a subgroup of $\mathbb{Z}^3$.

Solution: One shows that $(a, b, c) \mapsto a + 2b + 4c \pmod 8$ defines a homomorphism $\mathbb{Z}^3 \to \mathbb{Z}/8\mathbb{Z}$, whose kernel is H. (3 points)

b) Compute a basis of H.

Solution: Let $\pi_i : \mathbb{Z}^3 \to \mathbb{Z}$ be the projection on the ith coordinate. We have $\pi_3(H) = \mathbb{Z}$ since $(2, 1, 1) \in H$ and hence $1 \in \pi_3(H)$; a subgroup of $\mathbb{Z}$ containing 1 equals $\mathbb{Z}$. (2 points) By the lectures, $(2, 1, 1)$ together with a basis for $\ker(\pi_3) \cap H$ yields a basis of H. (2 points) By definition

$$\ker(\pi_3) \cap H = \{(a, b, 0) \mid a + 2b \equiv 0 \bmod 8\}\,.$$

We find $\pi_2(\ker(\pi_3) \cap H) = \mathbb{Z}$, because $(6,1,0) \in \ker(\pi_3) \cap H$ and $\pi_2(6,1,0) = 1$. (2 points) So a basis for $\ker(\pi_3) \cap H$ consists of $(6,1,0)$ together with a basis for $\ker(\pi_2) \cap \ker(\pi_3) \cap H$. (2 points) We have

$$\ker(\pi_2) \cap \ker(\pi_3) \cap H = \{(a,0,0) \mid a \equiv 0 \bmod 8\} = \mathbb{Z}(8,0,0),$$

(2 points) hence a basis of H is given by

$$((8,0,0), (6,1,0), (2,1,1).$$

(2 points)

c) Find the number of elements of $\mathbb{Z}^3/H$.

Solution: Let A denote the matrix with columns equal to the basis of H determined in b). Then either the order of $\mathbb{Z}^3/H$ is equal to the absolute value of the determinant of A or $\mathbb{Z}^3/H$ is infinite. (4 points) In this case the order is therefore 8 (1 point). Alternatively, this follows from a) using the first isomorphism theorem,

End of test (90 points)